k-FORCING NUMBER OF SOME WHEEL-RELATED GRAPHS

[1]Vincent L. Banot, [2*]Giselle Ann J. Alcoran-Alvarez, [3]Joel G. Adanza, [3]Michael P. Baldado Jr.

[1]College of Teacher Education, Negros Oriental State University, Kagawasan Avenue, Dumaguete City, Philippines

[2*]Physics and Geology Department, Negros Oriental State University, Kagawasan Avenue, Dumaguete City, Negros Oriental, Philippines

Email:giselleann.alvarez@norsu.edu.ph

[3]Mathematics Department, College of Arts and Sciences, Negros Oriental State University, Kagawasan Avenue, Dumaguete City, Philippines

ABSTRACT: *Let $G=(V,E)$ be a graph and k be a positive integer. A set $S(\subseteq V)$ is a k-forcing set if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule such that all of the vertices in G will eventually become colored. A colored vertex with at most k non-colored neighbors will cause each non-colored neighbor to become colored. The k-forcing number of G, denoted by $F_k(G)$, is the minimum cardinality of a k-forcing set.*

This study gave the k-forcing number of paths, cycles, fans, wheels, and wheel-related graphs.

Keywords: *k*-forcing number, sunflower, lotus-inside-circle, helm, gear, path, cycles, fans, wheels

1. INTRODUCTION

A subset S of vertices of a graph is a *k*-forcing set if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule until all the vertices will eventually become colored. A colored vertex with at most *k* non-colored neighbors will cause each non-colored neighbor to become colored. The *k*-forcing number of a graph, denoted by $F_k(G)$, is the cardinality of a smallest *k*-forcing set.

For example, consider graph *G* in Figure 1. Then $S_1 = \{a\}$ is a 2-forcing set, while $S_2 = \{b\}$ is not. The 2-forcing number of *G* is 1.

To see this, we note that *a* can 2-forces *b* and *f*, *b* can 2-forces *c* and *e*, *c* can 2-forces *d*. Hence, all the vertices of *G* will eventually be colored. Thus, $S_1 = \{a\}$ is a 2-forcing set.

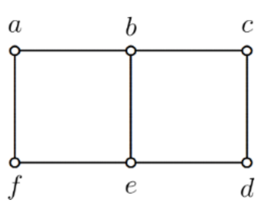

Figure 1. The graph *G*

On the other hand, we observe that *b* can not 2-force either *a*, *e* and *c*. Hence, color change cannot take effect. This shows that $S_2 = \{b\}$ is not a 2-forcing set.

Clearly, $S_1 = \{a\}$ is a minimum 2-forcing set. Thus, $F_2(G), = 1$.

The *k*-forcing concept is a generalization of the concept *zero forcing number* of a graph (the zero forcing number is actually the 1-forcing number). The concept was introduced by Barioli et al. [2] and independently, by Burgarth et al. [4]. These concepts were studied in [1–22].

2. PRELIMINARY RESULTS

In this section, we present some of the general properties of the *k*-forcing number. Clearly the *k*-forcing number of a graph cannot exceed its order. Also, if a graph is connected then its *k*-forcing number is less than its order. This observations are more formally stated in the next lemma and remark.

Lemma 2.1. *Let G be a graph of order n. Then $F_k(G) \leq n$ for all $k \in \mathbb{N}$.*

Remark 2.2. *If G is a connected graph of order n, then $F_k(G) < n$ for all $k \in \mathbb{N}$.*

The *Pigeon-hole Principle* may be stated as follows. If there are *n* boxes and there at most *n*-1 boxes, then at least one of the boxes is empty. Or equivalently, if box b_1 can hold at most j_1 marbles, box b_2 can hold at most j_2 marbles, so on, box b_n can hold at most j_n marbles, and there are less than $j_1 + j_2 + \cdots + j_n$ marbles, then at least one box is not full. The next lemma uses this idea.

Lemma 2.3. *Let $G_1, G_2, \ldots, G_r$ be connected graphs. Then $F_k(G_1 \cup G_2 \cup \cdots \cup G_n) = F_k(G_1) + F_k(G_2) + \cdots + F_k(G_n)$.*

Proof: Let $S_1, S_2, \ldots, S_n$ be minimum *k*-forcing sets of $G_1, G_2, \ldots, G_n$, respectively. Then clearly $S_1 \cup S_2 \cup \cdots \cup S_n$ is a *k*-forcing set of $G_1 \cup G_2 \cup \cdots \cup G_n$. Hence, $F_k(G_1 \cup G_2 \cup \cdots \cup G_n) \leq F_k(G_1) + F_k(G_2) + \cdots + F_k(G_n)$. Suppose that $F_k(G_1 \cup G_2 \cup \cdots \cup G_n) < F_k(G_1) + F_k(G_2) + \cdots + F_k(G_n)$. Let *S* be a minimum k-forcing set of $G_1 \cup G_2 \cup \cdots \cup G_n$. Then $|S| < F_k(G_1) + F_k(G_2) + \cdots + F_k(G_n)$. By the Pigeon-hole Principle, there exists at least one component, say *without loss of generality* G_1, with $V(G_1) \cap S < F_k(G_1)$. This implies that $V(G_1) \cap S$ is not a *k*-forcing set in G_1, that is, the color change rule can't cannot color all the vertices in G_1 by initially using the vertices in $V(G_1) \cap S$. This is a contradiction since S is a *k*-forcing set. Therefore, $F_k(G_1 \cup G_2 \cup \cdots \cup G_n) = F_k(G_1) + F_k(G_2) + \cdots + F_k(G_n)$. *QED*

The next remark follows from Remark 2.2 and Lemma 2.3.

Remark 2.4. *Let G be a graph of order n. If G has a non-trivial component, then $F_k(G) < n$.*

To see this, let $G = G_1 \cup G_2 \cup \cdots \cup G_k$, where G_i is a component for $i = 1, 2, \ldots, k$. Suppose that G_r is a non-trivial component. Then by Remark 2.2 $F_k(G_r) < |V(G_r)|$. Hence, by Lemma 2.3, $F_k(G) = F_k(G_1) + F_k(G_2) + \cdots +$

$F_k(G_r) + \cdots + F_k(G_k) < |V(G_1)| + |V(G_2)| + \cdots + |V(G_r)| + \cdots + |V(G_k)| = n$.

The *k*-forcing number is not less than the number of components. We present and show this idea in the next lemma.

Lemma 2.5. *Let G be a graph. If G have m components, then* $F_k(G) \geq m$*.*

Proof: Suppose that $F_k(G) < m$. Let *S* be a minimum *k*-forcing set. Then $|S| < m$. By the Pigeon-hole Principle, there exists at least one component, say *H*, with $V(H) \cap S = \emptyset$. This implies that the color change can't take place in *H*, that is, the vertices in *H* will remain uncolored. This is a contradiction since *S* is a *k*-forcing set. Therefore, $F_k(G) \geq m$. *QED*

The *k*-forcing number is precisely equal to the order when the graph is empty. We prove this conjecture in the next theorem.

Theorem 2.6. *Let G be a graph of order n > 1. Then* $F_k(G) = n$ *if and only if* $G = \overline{K_n}$*.*

Proof: Suppose that $F_k(G) = n$ and $G \neq \overline{K_n}$. If $G \neq \overline{K_n}$, then G has a non-trivial component. By Lemma 2.3, $F_k(G) < n$. This is a contradiction. Hence, if $F_k(G) = n$, then $G = \overline{K_n}$. Conversely, suppose that $G = \overline{K_n}$. Then *G* have *n* components. By Lemma 2.5, $F_k(G) \geq n$. Accordingly, $F_k(G) = n$. *QED*

Theorem 2.7 is due to Amos et al. (2015). We provided here an alternative proof.

The degree of a vertex is the number of incident edges. The greatest degree among the vertices is called the *maximum degree*, denoted by Δ. We present in Theorems 2.7, 2.8, and 2.9 some statements concerning the maximum degree Δ and the *k*-forcing number.

Theorem 2.7. *Let G = (V, E) be a connected graph. Then* $F_\Delta(G) = 1$.

Proof: Let $G = (V, E)$ be connected graph. If $|V| = 1$, then by Theorem 2.6 the theorem follows. So we assume that $|V| > 1$. Let $S = \{v\}$ with $v \in V$ and let $u \in V \setminus S$. Since *G* is connected, there exist a shortest path ($v = v_1, v_2, \ldots, v_{d(u,v)} = u$) in *G* connecting *u* and *v*. Since every vertex in *G* has at most Δ neighbors, v_1 can Δ-force v_2, v_2 can Δ-force v_3, . . ., v_{n-1} can Δ-force $v_{d(u,v)} = u$. Hence, *u* will eventually be colored. Since u is arbitrary, this implies all the vertices of *G* will eventually be colored. This shows that *S* is a Δ-forcing set. Therefore, $F_\Delta(G) = 1$. *QED*

Theorem 2.8. *Let G = (V, E) be a connected graph. If G is not Δ-regular, then* $F_{\Delta-1}(G) = 1$.

Proof: If *G* is not Δ-regular, then there exist $v \in V$ such that $deg_v(G) < \Delta$. Let $S = \{v\}$ and $u \in G \setminus S$. Since *G* is connected, there eexists a shortest path ($v = v_1, v_2, \ldots, v_{d(u,v)} = u$) in *G* connecting *v* and *u*. Note that, v_1 can (Δ−1)-force v_2, v_2 can (Δ−1)-force v_3, . . ., $v_{d(u,v)-1}$ can (Δ − 1)-force $v_{d(u,v)}$. Hence, *u* will eventually be colored. Since *u* is arbitrary, this implies all the vertices of *G* will eventually be colored. This shows that *S* is a Δ-1-forcing set. Therefore, $F_{\Delta-1}(G) = 1$. *QED*

Theorem 2.9. *Let G = (V, E) be a connected graph. If G is Δ-regular, then* $F_{\Delta-1}(G) = 2$.

Proof: Let $vw \in E$ and $S = \{v, w\}$. Let $u \in G \setminus S$. Since *G* is connected, there exist a shortest path ($v = v_1, v_2, \ldots, v_n = u$) in *G* connecting *v* and *u*. Note that, v_1 can (Δ − 1)-force v_2, v_2 can (Δ − 1)-force v_3, . . ., v_{n-1} can (Δ − 1)-force $v_n = u$. Hence, u will eventually be colored. Since *u* is arbitrary, this implies all the vertices of *G* will eventually be colored. Therefore, *S* is a (Δ − 1)-forcing set. Thus, $F_{\Delta-1}(G) \leq 2$.

Suppose $F_{\Delta-1}(G) = 1$. Let *S* be a minimum (Δ − 1)-forcing set, say $S = \{v\}$. Let $u \in N(S) \setminus S$. Since $deg_G(v) = \Delta$, *v* can not (Δ − 1) force *u*, that is, the color-change rule cannot take effect. This is a contradiction. Therefore, $F_{\Delta-1}(G) \geq 2$.

Accordingly, $F_{\Delta-1}(G) = 2$. *QED*

The following remark is clear, that is, a *k*-forcing set is also a *k*+1-forcing set. This idea is utilized in Corollary 2.11.

Remark 2.10. *Let G be a graph. Then every k-forcing set in G is also a k +1-forcing set.*

Corollary 2.11. *Let G be a graph. Then* $F_k(G) \geq F_{k+1}(G)$ *for all* $k \in \mathbb{N}$.

Proof : Let **A** be the set of all *k*-forcing set of *G* and $\mathscr{B}$ be the set of all *k* + 1-forcing set of *G*. By Remark 2.10, every *k*-forcing set in *G* is also a *k* + 1 forcing set. Hence, $\mathbf{A} \subseteq \mathscr{B}$. Thus, $\min\{|A| : A \in \mathbf{A}\} \geq \min\{|B| : B \in \mathscr{B}\}$, that is, $F_k(G) \geq F_{k+1}(G)$. *QED*

The minimum degree is the smallest degree among all vertices, denoted by $\delta(G)$. This notation is used in the next remark.

Remark 2.12. Let *G* be a connected graph. If *S* is a singleton *k*-forcing set of *G*, then $\delta(G) \leq k$.

3. *k*-FORCING NUMBER OF PATHS

In this section we give *k*-forcing number of the family of paths.

Theorem 3.1. *Let* P_n *be a path of order n > 1. Then* $F_k(P_n) = 1$ *for all positive integer k.*

Proof: Let $P_n = (1, 2, \ldots, n)$ be a path of order n and $S = \{1\}$. Note that 1 can 1-force 2, 2 can 1-force 3, . . ., *n*-1 can 1-force *n*. Hence, *S* is a 1-forcing set. Clearly, *S* is a minimum 1-forcing set. Thus, $F_1(G) = 1$.

Finally by Corollary 2.11, $F_k(P_n) = 1$ for all positive integer *k*. *QED*

Theorem 3.2. *Let G be a non-trivial graph of order n. Then* $F_1(G) = 1$ *if and only if* $G = P_n$.

Proof: Suppose that $F_1(G) = 1$. Let $S = \{u_1\}$ be a 1-forcing set of *G*. Since $S = \{u_1\}$ is a 1-forcing set, we must have $\delta(u_1) = 1$, that is u_1 is a leaf. Now, let $N(u_1) = \{u_2\}$. Then by the color-change rule, u_1 can 1-force u_2. Next, $\delta(u_2)$ must be equal to 2, since S is a 1-forcing set. Now, let $N(u_2)\backslash\{u_1\} = \{u_3\}$. Then by the color-change rule, u_2 can 1-force u_3. Note again that $\delta(u_3)$ must be equal to 2, since S is a 1-forcing set. Now, let $N(u_3)\backslash\{u_2\} = \{u_4\}$. Then by the color-change rule, u_3 can 1-force u_4. Continuing in this fashion, the color change action will eventually terminate with u_{n-1}1-forcing u_n. Thus, the degree sequence of G is $\langle 1, 2, 2, \dots, 1\rangle$, that is, G is the path P_n.

Conversely, if G is a path, then by Theorem 3.2 $F_1(G)$ = 1. *QED*

4. *k*-FORCING NUMBER OF CYCLES

In this section we give *k*-forcing number of the family of cycle graphs,

Theorem 4.1. *Let C_n be a cycle of order n. Then $F_1(C_n)$ = 2.*

Proof: Let $C_n = [1, 2, \dots, n]$ be a cycle of order *n*, and $S = \{1, 2\}$. Consider the following cases.

Case 1. *n* is even

If *n* is even, then: 2 can 1-force 3, and 1 can 1-force *n*; 3 can 1-force 4, and *n* can 1-force *n* − 1; 4 can 1-force 5, and *n* − 1 can 1-force *n* − 2; and so on. Until eventually, *n*/2 can 1-force (*n* + 2)/2, and (*n* + 6)/2 can 1-force (*n* + 4)/2.

Case 2. *n* is odd

If *n* is odd, then: 2 can 1-force 3, and 1 can 1-force *n*; 3 can 1-force 4, and *n* can 1-force *n* − 1; 4 can 1-force 5, and *n* − 1 can 1-force *n* − 2; and so on. Until eventually, (*n* − 1)/2 can 1-force $\lceil n/2 \rceil$, and (n + 3)/2 can 1-force $\lceil (n+4)/2 \rceil$.

In any case, all the vertices of C_n will eventually be colored. Hence, *S* is a 1-forcing set. Thus, $F_1(C_n) \leq 2$. Note that a 1-forcing set in C_n cannot be singleton. Therefore, $F_1(C_n)$ = 2. *QED*

Theorem 4.2. *Let C_n be a cycle of order n. Then $F_k(C_n)$ = 1 for all positive integer k ≥ 2.*

Proof: Let $C_n = [1,2,\dots,n]$ be a cycle of order *n*. Let $S = \{1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then: 1 can 2-force 2 and n; 2 can 2-force 3, and *n* can 2-force *n*-1; 3 can 2-force 4, and *n*-1 can 2-force *n*-2; and so on. Until eventually, (*n* − 2)/2 can 2-force *n*/2, and (*n* + 4)/2 can 2-force (*n* + 2)/2.

Case 2. *n* is odd

If *n* is odd, then: 1 can 2-force 2 and *n*; 2 can 2-force 3, and *n* can 2-force *n*-1; 3 can 2-force 4, and *n*-1 can 2-force *n*-2; and so on. Until eventually, (*n* − 1)/2 can 2-force $\lceil n/2 \rceil$, and (*n* + 3)/2 can 2-force $\lceil (n+4)/2 \rceil$.

In any case, all the vertices of C_n will eventually be colored. Hence, *S* is a 2-forcing set. Thus, $F_2(C_n)$ = 1. By Corollary 2.11, $F_k(C_n)$ = 1 for all positive integer $k \geq 2$. *QED*

5. *k*-FORCING NUMBER OF FANS

In this section we give *k*-forcing number of the family of fan graphs.

Theorem 5.1. *Let F_n be a fan of order n* + 1. *Then* $F_1(F_n)$ = 2.

Proof: Let $F_n = (\{0\}, \varnothing) + [1,2,\dots,n]$ be a fan of order *n* + 1. Let $S = \{0, 1\}$. Then 1 can 1-force 2, 2 can 1-force 3,…, *n* − 1 can 1-force *n*. Hence, all the vertices of F_n are eventually colored. Thus, *S* is a 1-forcing set. Note that $F_1(F_n)$ cannot be 1. Therefore, $F_1(F_n)$ = 2. *QED*

Theorem 5.2. *Let F_n be a fan of order n* + 1. *Then $F_k(F_n)$ =* 1 *for all positive integer k* > 1.

Proof: Let *n* > 2 be a positive integer and $F_n = (\{0\}, \varnothing) + [1, 2,\dots, n]$ be a fan of order *n* + 1. Let $S = \{1\}$. Then 1 can 2-force 2 and 0, 2 can 2-force 3, . . ., *n* − 1 can 2-force *n*. Hence, all the vertices of F_n are eventually colored. Thus, *S* is a 2-forcing set. Therefore, $F_2(F_n)$ = 1. By Corollary 2.11, $F_k(F_n)$ = 1 for all positive integer *k* > 1. *QED*

6. *k*-FORCING NUMBER OF WHEELS

In this section we give *k*-forcing number of the family of wheel graphs,

Theorem 6.1. *Let W_n be a wheel of order n* + 1. *Then $F_2(W_n)$* = 2.

Proof: Let $W_n = (\{0\}, \varnothing) + [1, 2, \dots, n]$ be a wheel of order *n* + 1. Let $S = \{0, 1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then: 1 can 2-force 2 and n; 2 can 2-force 3, and *n* can 2-force *n* − 1; 3 can 2-force 4, and *n* − 1 can 2-force *n* − 2; and so on. Until eventually, (*n* − 2)/2 can 2-force *n*/2, and (*n* + 4)/2 can 2-force (*n* + 2)/2.

Case 2. *n* is odd

If *n* is odd, then: 1 can 2-force 2 and *n*; 2 can 2-force 3, and *n* can 2-force *n*−1; 3 can 2-force 4, and *n* − 1 can 2-force *n* − 2; and so on. Until eventually, (*n* − 1)/2 can 2-force $\lceil n/2 \rceil$, and (n + 3)/2 can 2-force $\lceil (n+4)/2 \rceil$.

In any case, all the vertices of W_n will eventually be colored. Hence, *S* is a 2-forcing set. Note that $F_2(W_n)$ cannot be 1. Thus, $F_2(W_n)$ = 2. *QED*

Theorem 6.2. *Let W_n be a wheel of order n* + 1. *Then $F_k(W_n)$* = 1 *for all positive integer k* ≥ 3.

Proof: Let $W_n = (\{0\}, \varnothing) + [1, 2,\dots, n]$ be a wheel of order *n* + 1. Let $S = \{1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then: 1 can 3-force 0, 2 and *n*; 2 can 3-force 3, and *n* can 3-force $n-1$; 3 can 3-force 4, and $n-1$ can 3-force $n-2$; and so on. Until eventually, $(n-2)/2$ can 3-force $n/2$, and $(n+4)/2$ can 3-force $(n+2)/2$.

Case 2. *n* is odd

If *n* is odd, then: 1 can 3-force 0, 2 and *n*; 2 can 3-force 3, and *n* can 3-force $n-1$; 3 can 3-force 4, and $n-1$ can 3-force $n-2$; and so on. Until eventually, $(n-1)/2$ can 3-force $\lceil n/2 \rceil$, and $(n+3)/2$ can 3-force $\lceil (n+4)/2 \rceil$.

In any case, all the vertices of W_n will eventually be colored. Hence, *S* is a 3-forcing set. Thus, $F_3(W_n) = 1$. By Corollary 2.11, $F_k(W_n) = 1$ for all positive integer $k \geq 3$. *QED*

7. *k*-FORCING NUMBER OF WHEEL-RELATED GRAPHS

In this section we give *k*-forcing number of the family of sunflower graphs, the family of sunflower graphs, the family of lotus inside circle graphs, the family of helm graphs, and the family of gear graphs.

Let $W_n = (\{v\}, \varnothing) + [v_1, v_2, \ldots, v_n]$ be a wheel of order $n+1$. The *sunflower graph* SF_n is the graph of order $2n+1$ obtained from W_n by adding vertices w_i joined by edges to vertices v_i and $v_{i+1(\mathrm{mod}\, n)}$ for $i = 1, 2, \ldots, n$. Figure 2 presents the gear graph of order 17.

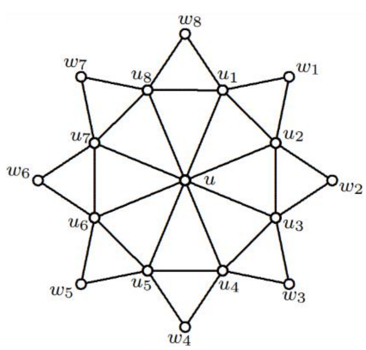

Figure 2. The sunflower graph of order 17

Theorem 7.1. *Let* SF_n *be the sunflower graph of order* $2n+1$. If $k = 2$, then $F_k(SF_n) = 2$.

Proof: Let SF_n be the sunflower graph of order $2n+1$ obtained from $W_n = (\{\mathrm{v}\}, \varnothing) + [v_1, v_1, \ldots, v_n]$ by adding vertices u_i joined by edges to vertices v_i and v_{i+1} (mod_n) for $i = 1, 2, \ldots, n$. Let $S = \{v, u_1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then: u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 2-force u_2 and v_3; v_n can 2-force u_{n-1} and $v_{n/2}$, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 2-force $u_{(n+2)/2}$, and $v_{n/2}$ can 2-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is odd, then: u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 2-force u_2 and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 2-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 2-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$.

In any case, all the vertices of SF_n will eventually be colored. Hence, *S* is a 2-forcing set. Thus, $F_2(SF_n) \leq 2$. By Remark 2.12 a 2-forcing set of SF_n cannot be singleton. Therefore, $F_2(SF_n) = 2$. *QED*

Theorem 7.2. *Let* SF_n *be the sunflower graph of order* $2n+1$. *If* $k \geq 3$, *then* $F_k(SF_n) = 1$.

Proof: Let SF_n be the sunflower graph of order $2n+1$ obtained from $W_n = (\{v\}, \varnothing) + [v_1, v_1, \ldots, v_n]$ by adding vertices u_i joined by edges to vertices v_i and $v_{i+1(mod\ n)}$ for $i = 1, 2, \ldots, n$. Let $S = \{u_1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then: u_1 can 3-force v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2, v_3, and v; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force $u_3 =$ and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 3-force $u_{(n+2)/2}$, and $v_{n/2}$ can 3-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is odd, then: u_1 can 3-force v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2, v_3, and v; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 3-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 3-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$.

In any case, all the vertices of SF_n will eventually be colored. Hence, *S* is a 3-forcing set. Thus, $F_2(SF_n) = 1$. By Corollary 2.11, $F_k(SF_n) = 1$ for all positive integer $k \geq 3$. *QED*

Let $C_n = [w_1, w_2, \ldots, w_n]$ be a cycle of order *n*, and $K_{1,n} = (\{v\}, \varnothing) + (\{v_1, v_2, \ldots, v_n\}, \varnothing)$ be a star of order $n+1$. The *graph lotus inside circle*, denoted by LC_n, is the graph of order $2n+1$ obtained from C_n and $K_{1,n}$ by joining each vertex u_i to w_i and $w_{i+1(\mathrm{mod}\, n)}$ for $i = 1, 2, \ldots, n$. Figure 3 presents the lotus inside circle graph of order 17.

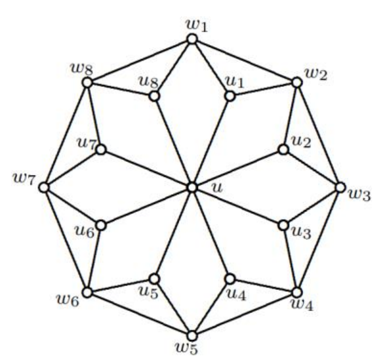

Figure 3. The lotus inside circle graph of order 17

Theorem 7.3. *Let* LC_n *be the lotus-inside-circle graph of order* $2n+1$. *If* $k = 2$, *then* $F_k(LC_n) = 2$.

Proof: Let $C_n = [v_1, v_1, \ldots, v_n]$ be a cycle of order *n*, and $K_{1,n} = (\{u\}, \varnothing) + (\{u_1, u_2, \ldots, u_n\}, \emptyset)$ be a star of order *n* + 1. Let LC_n be the lotus inside circle graph of order 2*n* + 1 obtained from C_n and $K_{1,n}$ by joining each vertex u_i to vertices vi and $v_{i+1(mod\ n)}$ for *i* = 1, 2, . . . , *n*.. Let S = {*v*, u_1} and consider the following cases.

Case 1. *n* is even

If *n* is even, then: u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 2-force u_2 and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 2-force $u_{(n+2)/2}$, and $v_{n/2}$ can 2-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is odd, then: u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 2-force u_2, and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 2-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 2-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$

In any case, all the vertices of LC_n will eventually be colored. Hence, *S* is a 2-forcing set. Thus, $F_2(LC_n) \leq 2$. By Remark 2.12 a 2-forcing set of SF_n cannot be singleton. Therefore, $F_2(LC_n) = 2$. *QED*

Theorem 7.4. *Let* LC_n *be the lotus-inside-circle graph of order* 2*n* + 1. *If* $k \geq 3$, *then* $F_k(LC_n) = 1$.

Proof: Let $C_n = [v_1, v_1, \ldots, v_n]$ be a cycle of order n, and $K_{1,n} = (\{u\}, \varnothing) + (\{u_1, u_2, \ldots, u_n\}, \emptyset)$ be a star of order *n* + 1. Let LC_n be the lotus inside circle graph of order 2*n* + 1 obtained from C_n and $K_{1,n}$ by joining each vertex u_i to vertices v_i and $v_{i+1(mod\ n)}$ for *i* = 1, 2, . . . , *n*. Let S = {u_1} and consider the following cases.

Case 1. *n* is even

If *n* is even, then: u_1 can 3-force *v*, v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2 and v_3; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 3-force $u_{(n+2)/2}$, and $v_{n/2}$ can 3-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is odd, then: u_1 can 3-force *v*, v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2 and v_3; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 3-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 3-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$.

In any case, all the vertices of LC_n will eventually be colored. Hence, *S* is a 3-forcing set. Thus, $F_3(LC_n) = 1$. By Corollary 2.11, $F_k(LC_n)$= 1 for all positive integer $k \geq 3$.

QED

Let $W_n = (\{v\}, \varnothing) + [v_1, v_2, \ldots, v_n]$ be a wheel of order $n+1$. The *helm* H_n is the graph of order $2n+1$ obtained from W_n by attaching pendant edges for $v_i w_i$ for $i = 1, 2, \ldots, n$. Figure 4 presents the helm graph of order 17.

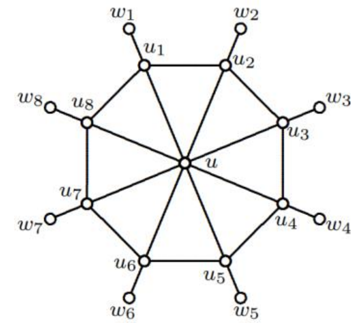

Figure 4. The helm graph of order 17

Theorem 7.5. *Let* H_n *be the helm graph of order* 2*n* + 1. If *k* = 2, *then* $F_k(H_n) = 2$.

Proof: Let H_n be the helm graph of order 2*n* + 1 obtained from the wheel $W_n = (\{v\}, \varnothing) + [v_1, v_1, \ldots, v_n]$ by attaching pendant edges $v_i u_i$ for *i* = *1*, *2*, . . . , *n*. Let *S* = {*v*, u_1} and consider the following cases.

Case 1. *n* is even

If *n* is even, then: u_1 can 2-force v_1; $v_1$can 2-force v_n and v_2; v_2 can 2-force u_2 and v_3, and v_n can 2-force u_n and v_{n-1}; v_3 can 2-force u_3 and v_4, and v_{n-1} can 2-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{(n+2)/2}$ can 2-force $u_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is even, then: u_1 can 2-force v_1; v_1 can 2-force v_n and v_2; v_2 can 2-force u_2 and v_3, and v_n can 2-force u_n and v_{n-1}; v_3 can 2-force u_3 and v_4, and v_{n-1} can 2-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 2-force $u_{\lceil n/2 \rceil}$.

In any case, all the vertices of H_n will eventually be colored. Hence, *S* is a 2-forcing set. Thus, $F_2(H_n) \leq 2$. By Remark 2.12 a 2-forcing set of H_n cannot be singleton. Therefore, $F_2(H_n) = 2$. *QED*

Theorem 7.6. *Let* H_n *be the helm graph of order* 2*n* + 1. *If* $k \geq 3$, *then* $F_k(H_n) = 1$.

Proof: Let H_n be the helm graph of order 2*n* + 1 obtained from the wheel $W_n = (\{v\}, \varnothing) + [v_1, v_2, \ldots, v_n]$ by attaching pendant edges $v_i u_i$ for *i* = 1, 2, . . . , *n*. Let *S* = {u_1} and consider the following cases.

Case 1. *n* is even

If *n* is even, then: u_1 can 3-force v_1; v_1 can 3-force *v*, v_n and v_2; v_2 can 3-force u_2 and v_3, and v_n can 3-force u_n and v_{n-1}; v_3 can 3-force u_3 and v_4, and v_{n-1} can 3-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{(n+2)/2}$ can 3-force $u_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is even, then: u_1 can 3-force v_1; v_1 can 3-force *v*, v_n and v_2; v_2 can 3-force u_2 and v_3, and v_n can 3-force u_n and v_{n-1}; v_3 can 3-force u_3 and v_4, and v_{n-1} can 3-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 3-force $u_{\lceil n/2 \rceil}$.

In any case, all the vertices of H_n will eventually be colored. Hence, S is a 3-forcing set. Thus, $F_3(H_n) = 1$. By Corollary 2.11, $F_k(H_n) = 1$ for all positive integer $k \geq 3$. *QED*

Let $W_n = (\{u\}, \varnothing) + [u_1, u_2, \ldots, u_n]$ be a wheel of order $n+1$. The *gear* graph G_n is the graph obtained from W_n by adding vertices w_i in between adjacent vertices vertices u_i and $u_{i+1(\text{mod}\, n)}$ for $i = 1, 2, \ldots, n$. Figure 5 presents the gear graph of order 17.

Theorem 7.7 gives the zero-forcing number of gear graph.

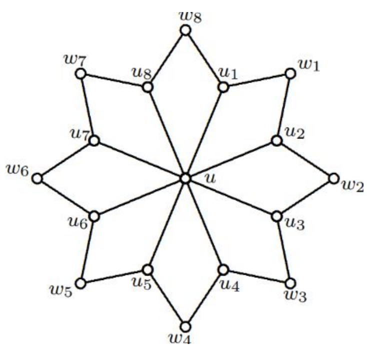

Figure 5. The gear graph of order 17

Theorem 7.7. *Let* G_n *be the gear graph of order* $2n + 1$. *If* $k = 1$, then $F_k(G_n) = 3$.

Proof: Let G_n be the gear graph obtained from $W_n = (\{v\}, \varnothing) + [v_1, v_1, \ldots, v_n]$ by adding vertices u_i in between adjacent vertices v_i and $v_{i+1(mod\ n)}$ for $i = 1, 2, \ldots, n$. Let $S = \{v, v_1, u_1\}$ and consider the following cases.

Case 1. *n* is even

If n is even, then: u_1 can 1-force v_2, and v_1 can 1-force u_n; v_2 can 1-force u_2, and u_n can 1-force v_n; u_2 can 1-force v_3, and v_n can 1-force u_{n-1}; and so on. Until eventually, $v_{(n+2)/2}$ can 1-force $u_{(n+2)/2}$.

Case 2. *n* is odd

If n is odd, then: u_1 can 1-force v_2, and v_1 can 1-force u_n; v_2 can 1-force u_2, and u_n can 1-force v_n; u_2 can 1-force v_3, and v_n can 1-force u_{n-1}; and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 1-force $u_{\lceil n/2 \rceil}$.

In any case, all the vertices of G_n will eventually be colored. Hence, S is a 1-forcing set. Thus, $F_1(G_n) \leq 3$. Note that a 1-forcing set of G_n cannot have less than 3 elements. Therefore, $F_1(G_n) = 3$. *QED*

Theorem 7.8 gives the *k*-forcing number of gear graph for $k \geq 2$.

Theorem 7.8. *Let* G_n *be the gear graph of order* $2n + 1$. *If* $k \geq 2$, *then* $F_k(G_n) = 1$.

Proof: Let G_n be the gear graph obtained from $W_n = (\{v\}, \varnothing) + [v_1, v_1, \ldots, v_n]$ by adding vertices u_i in between adjacent vertices v_i and $v_{i+1(mod\ n)}$ for $i = 1, 2, \ldots, n$. Let $S = \{v, u_1\}$ and consider the following cases.

Case 1. *n* is even

If n is even, then: u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v, and v_2 can 2-force u_2; u_n can 2-force v_n, and v_3 can 2-force u_3; and so on. Until eventually, $v_{(n+2)/2}$ can 2-force $u_{(n+2)/2}$.

Case 2. *n* is odd

If n is odd, then: u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 2-force u_2 and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 2-force $u_{\lceil n/2 \rceil}$.

In any case, all the vertices of G_n will eventually be colored. Hence, S is a 2-forcing set. Thus, $F_2(G_n) = 1$. By Corollary 2.11, $F_k(G_n) = 1$ for all positive integer $k \geq 2$. *QED*

8. ACKNOWLEDGMENT

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